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DISTANCE CLOSED RESTRAINED DOMINATION CRITICAL GRAPHS

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ABSTRACT: In a graph G = (V, E), a set S$\subset$V(G) is said to be a distance closed restrained dominating set if (i) <S> is distance closed and (ii) S is a restrained dominating set. The cardinality of the minimum distance closed restrained dominating set is called the *distance closed restrained domination number* and it is denoted by $\gamma_{rdcl}(G)$. In this paper, we discuss the critical concept in distance closed restrained domination which deals with those graphs that are critical in the sense that their distance closed restrained domination number drops when any missing edge is added. Also, we analyze some structural properties of those distance closed restrained domination critical graphs.

KEYWORDS: domination number, distance, eccentricity, radius, diameter, self-centered graph, neighborhood, induced sub graph, distance closed restrained dominating set, distance closed restrained domination critical graph.

1. Introduction

Graphs discussed in this paper are connected and simple graphs only. For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $deg_G(v)$ and minimum degree and maximum degree of G are indicated by $\delta(G)$ and $\Delta(G)$ respectively. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. For a connected graph G, the *eccentricity* $e_G(v) = \max \{d_G(u, v): \forall u \in V(G)\}$ and the eccentric set $E_G(v) = \{u \in V(G): d(v, u) = e_G(v)\}$. If there is no confusion, we simply use the notation deg (v), d(u, v), e(v) and E(v) to denote degree, distance, eccentricity and eccentric set respectively for a connected graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted by r(G) and d(G) respectively. If these two are equal in a graph, that graph is called *self-centered* graph with radius r and is called an *r self-centered* graph. Such graphs are 2-connected graphs. A vertex u is said to be an eccentric vertex of v in a graph G, if d(u, v) = e(v) in that graph. For v$\in$V(G), the *neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood* of v. One important aspect of the concept of distance and eccentricity is the existence of polynomial time algorithm to analyze them. The concept of distance and related properties are studied in [2]. The new concepts such as ideal sets, distance preserving sub graphs, eccentricity preserving sub graphs, super eccentric graph of a graph, pseudo geodetic graphs are introduced and structural properties of those graphs are studied in [6].

The concept of domination in graphs was introduced by Ore [10] in 1962. It is originated from the chess game theory which paved the way to the development of the study of various domination parameters and then relation to various other graph parameters. A set D$\subseteq$V(G) is called a dominating set of G if every vertex in V(G) – D is adjacent to some vertex in D and D is said to be a minimal dominating set if D – {v} is not a dominating set for any v$\in$D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on dominating sets. A dominating set D is called a restrained dominating set if every vertex in V – D is adjacent to a vertex in D as well as a vertex in V – D. Also, the cardinality of a minimum restrained dominating set of G is called the restrained domination number of G and is denoted by $\gamma_r(G)$. The concept of dominating set and different types of dominating set are studied in [3] and [4].

Graphs which are critical with respect to a given property frequently play an important role in the investigation of that property. Not only are such graphs of considerable interest in their own right, but also a knowledge of their structure often aids in the development of general theory. In particular, when investigating any finite structure, a great number of results are proven by induction. Consequently, it is desirable to learn as much as possible about those graphs that are critical with respect to a given property so as to aid and abet such investigation. A graph G is said to be domination critical if for every edge $e \notin E(G)$, $\gamma(G + e) < \gamma(G)$. If G is a domination critical graph with $\gamma(G) = k$, we will say G is k-domination critical or just k-critical. The 1-critical graphs are K_n, for $n \geq 1$. The concept of domination critical graphs and their structural properties are studied in [1], [5] and [11].

The critical concept in graphs plays an important role in the study of structural properties of graphs and hence it will be useful to study any communication model. In this paper, we introduced a new domination critical graph called distance closed restrained domination critical graphs through which the structural properties of those graphs are studied.

2. Prior Results

The concept of ideal set is defined and studied in the doctoral thesis of Janakiraman [6] and the concept of ideal sets in graph theory is due to the related concept of ideals in ring theory in algebra. The ideals in a ring are defined with respect to the multiplicative closure property with the elements of that ring. Similarly, the ideal set in a graph is defined with respect to the distance property between the ideal set and the vertices of the graph. Thus, the ideal set of a graph G is defined as follows:

Let I be a vertex subset of G. Then I is said to be an *ideal set* of G if

(i) For each vertex $u \in I$ and for each $w \in V - I$, there exists at least one vertex $v \in I$ such that $d_{<I>}(u, v) = d_G(u, w)$.

(ii) I is the minimal set satisfying (i).

Also, a graph G is said to be a 0-ideal graph if it has no non-trivial ideal set other than G. The ideal set without the minimality condition is taken as a distance closed set in the present work. Thus, the distance closed dominating set of a graph G is defined as follows:

A subset $S \subseteq V(G)$ is said to be a *distance closed dominating (D.C.D) set*, if

(i) $<S>$ is distance closed;

(ii) S is a dominating set.

The cardinality of a minimum D.C.D set of G is called the *distance closed domination number* of G and is denoted by γ_{dcl}. Clearly from the definition, $1 \leq \gamma_{dcl} \leq p$ and graph with $\gamma_{dcl} = p$ is called a 0-distance closed dominating graph. The definition and the extensive study of the above said distance closed domination in graphs and the structural properties of distance closed domination critical graphs are studied in [7], [8] and [9]. The following results given in [8] and [9] are used to prove many results in the present work.

Theorem 2.1 [8]: G is 3-D.C.D critical if and only if

(i) G is connected.
(ii) G has $\gamma_{dcl}(G) = 3$.
(iii) G has exactly one vertex with eccentricity equal to 1.
(iv) For every pair of non-adjacent vertices at least one of them is of degree $p - 2$.

Theorem 2.2 [8]: A graph G is 4-D.C.D critical if and only if

(i) G is connected.
(ii) G has $\gamma_{dcl}(G) = 4$.
(iii) For any two non-adjacent vertices at least one of them is of degree $p - 2$.

Theorem 2.3 [9]: If G is any k-D.C.D type (I) critical graph then

(i) G is of diameter $k - 1$ and radius $k - \frac{1}{2}$.
(ii) G has at most two pendant vertices.
(iii) G can have at most $(k - 2)$ cut vertices.
(iv) G is diameter edge (addition) critical.

(v) G is also a block and is Hamiltonian.

Theorem 2.4 [9]: If G is any k-D.C.D type (II) critical graph then

(i) G is $\frac{k}{2}$-self-centered.
(ii) G is a block.
(iii) G is radius edge (addition) critical.
(iv) G is Hamiltonian.

3. Main Results

Continuing the above, we studied the critical concept of the distance closed restrained domination in graphs while adding an edge in that graph. The distance closed restrained domination critical graph is defined as follows.

A graph G is said to be a *distance closed restrained domination critical* if for every edge $e \notin E(G)$, $\gamma_{rdcl}(G + e) < \gamma_{rdcl}(G)$. If G is a D.C.R.D critical graph with $\gamma_{rdcl}(G) = k$, then G is said to be k-D.C.R.D critical. In a graph G, addition of an edge $e \notin E(G)$ will reduce the distance closed restrained domination number of G implies that it will reduce the distance closed domination number or the restrained domination number of G. Thus, the critical concept of the distance closed restrained domination can be analyzed in the following two ways.

(i) D.C.R.D critical graphs with respect to distance closed domination property.
(ii) D.C.R.D critical graphs due to restrained property.

3.1 D.C.R.D critical graphs with respect to distance closed domination property

If a graph G is k-D.C.R.D critical with respect to distance closed domination property, then G must be k-D.C.D critical and also to maintain the restrained property, some extra conditions are added on it. Hence, initially a k-D.C.D critical graph is considered and then the extra conditions are added one by one to maintain the restrained property in both G and (G + e), $e \notin E(G)$. The following results are trivial from the k-D.C.D critical graphs.

Proposition 3.1: For any k-D.C.R.D critical graph G, $|V(G)| \geq k + 2$.
Proposition 3.2: There is no 2-D.C.R.D critical graph.

3.1.1 Structural properties of k-D.C.R.D, k ≤ 6 type (I) and type (II) critical graphs

Any k-D.C.D critical graph G with type (I) (structure having every minimum D.C.D set is a path of length k) and type (II) (structure having every minimum D.C.D set is a cycle of length k) are k-D.C.R.D critical with some imposed conditions. Also, any k-D.C.R.D critical graph is k-D.C.D critical. Hence, in most of the cases both D.C.R.D critical graphs and D.C.D critical graphs have the same structural properties.

Theorem 3.1: If G is a 3-D.C.D critical graph with $p \geq 5$ and having a cut vertex, then G cannot be 3-D.C.R.D critical.

Proof: Let G be a 3-D.C.D critical graph with a cut vertex. Then $\delta(G) = 1$ and G has exactly one vertex of degree $(p - 1)$, say v. Then, v must be a cut vertex of G and the vertex of degree 1, say u is adjacent to only the vertex v. Also, every D.C.R.D set of G must contain this vertex u (by Theorem 4.2.1). That is, the set of vertices {u, v, w}, where $d(w) = p - 2$ forms a D.C.R.D set for G.

Claim: ***u is the only vertex with degree less than or equal to p – 3 in G***

Suppose that, if G has more than one vertex of degree less than or equal to p – 3, then u must be adjacent to all these vertices (as the set of vertices with degree less than or equal to p – 3 forms a clique). Hence, $d(u) \geq 2$, a contradiction and hence u is the only vertex with degree less than or equal to p – 3 in G.

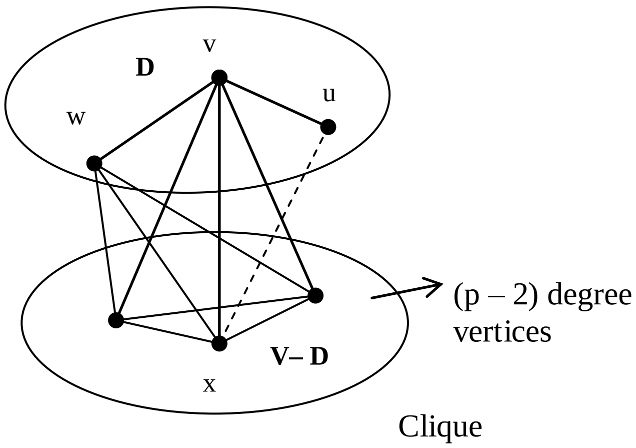

Figure 3.1 - *Structure of a 3-D.C.R.D critical graph with a cut vertex*

Thus, every vertex x in V – D is of degree p – 2 and also <V – D> is a clique (as they are all non-adjacent to u). Now, if we add an edge between u to any vertex x in V – D, then the set D^1 = {v, x} forms a D.C.D set for (G + ux) and u is adjacent to only those two vertices of D^1 in (G + ux) (refer Figure 4.2). Hence, D^1 cannot be a D.C.R.D set of (G + ux), for every pair of non-adjacent vertices u and x of G and hence G cannot be 3-D.C.R.D critical.

Theorem 3.2: Any 3-D.C.D critical graph, which is also a block, is 3-D.C.R.D critical.

Proof: Let G be a 3-D.C.D critical graph without any cut vertex. Then $\delta(G) \geq 2$ and G has exactly one vertex of degree (p – 1), say v.

Claim: *δ(G) cannot be equal to 2*

Let u be a vertex of degree 2 in G. Then, u must be adjacent to v and it is also adjacent to a vertex w in V(G) – {u, v}. Also, every vertex in V(G) – {u, v, w} is of degree equal to p – 2. For otherwise, if a vertex x in V(G) – {u, v, w} is of degree less than or equal to p – 3, then u must be adjacent to x, a contradiction to d(u) = 2. Therefore, d(x) = p – 2, for every x in V(G) – {u, v, w}. Hence, < V(G) – {u, v, w}> is a clique and also every vertex in V(G) – {u, v, w} must be adjacent to w and hence d(w) = p – 1, a contradiction to G is 3-D.C.D critical. Therefore, $\delta(G) \geq 3$.

Since G is 3-D.C.D critical, $\gamma_{dcl}(G + xy) = 2$, for every pair of non-adjacent vertices (x, y) in G. Also, $\delta(G) \geq 3$ implies that every vertex in G is adjacent to at least one vertex of V – D, where D is any D.C.D set of (G + xy). Hence, every D.C.D set D of G is a D.C.R.D set of G and hence G is 3-D.C.R.D critical.

Remark 3.1: As any 3-D.C.D critical graph without cut vertex is 3-D.C.R.D critical, we have the following theorems without proof.

Theorem 3.3: If G is a 3-D.C.R.D critical graph, then
- (i) G has exactly one vertex with eccentricity equal to 1.
- (ii) For every pair of non-adjacent vertices at least one of them is of degree p – 2.
- (iii) $\Delta(G) = p - 1$ and $3 \leq \delta(G) \leq p - 2$.

Theorem 3.4: If G is a 3-D.C.R.D critical graph, then G is of diameter equal to two.

Proposition 3.3: If G is a 3-D.C.R.D critical graph, then any minimum D.C.R.D set of G can have at most one vertex of degree less than or equal to (p – 3).

Theorem 3.5: There exists no graph G for which both G and $\overline{G}$ are 3-D.C.R.D critical graphs.

Theorem 3.6: If G is a 4-D.C.D critical graph with $p \geq 6$, then G is 4-D.C.R.D critical.

Proof: Let G be a 4-D.C.D critical graph with $p \geq 6$. Then by Theorem 3.3.1, for every pair of non-adjacent vertices of G at least one of them is of degree $p - 2$. Also, any minimum D.C.R.D set D of G can have at most two vertices with degree less than or equal to $p - 3$ and <D> is a cycle. Since $p \geq 6$, any vertex u in V(G) with degree $p - 2$ must be of degree greater than or equal to 4. Also, any vertex u in V – D with degree less than or equal to $p - 3$ must be of degree greater than or equal to 4 (as it is adjacent to all the 4 vertices of D). Hence, it is enough to consider only the vertices v with degree less than or equal to $p - 3$ in D and $d(v) \leq 3$. Therefore, we have the following two cases.

Case (i): D has exactly one vertex v with $d(v) \leq 3$

If v is the only vertex of degree less than or equal to $p - 3$ in D, then D = {x, v, y, z}, where $d(x) = d(y) = d(z) = p - 2$. Since $d(v) \leq 3$, v is not adjacent to at least one vertex of V – D. Now, if we add an edge between any pair of non-adjacent vertices x^1 and y^1 of G, then we can have a D.C.R.D set D^1 for $(G + x^1y^1)$ that contains the vertex v with $|D^1| = 3$.

Case (ii): D has two vertices v and w such that $d(v) \leq 3$, $d(w) \leq 3$

If v and w are the two vertices of degree less than or equal to $p - 3$ in D, then D = {v, w, x, y}, where $d(x) = d(y) = p - 2$ forms a D.C.D set for G. Since $d(v) \leq 3$ and $d(w) \leq 3$, they each non-adjacent at least one vertex of V – D. Now, if we add an edge between any pair of non-adjacent vertices x^1 and y^1 of G, then we can have a D.C.R.D set D^1 for $(G + x^1y^1)$ that contains at least one of the vertices of {v, w} with $|D^1| = 3$. Hence the proof.

Remark 3.2: As any 4-D.C.D critical graph is 4-D.C.R.D critical, we have the following theorems without proof.

Theorem 3.7: If G is a 4-D.C.R.D critical graph, then

(i) For every pair of non-adjacent vertices at least one of them is of degree $p - 2$.

(ii) $3 \leq \delta(G) \leq p - 2$ and $\Delta(G) = p - 2$.

Proposition 3.4: If G is a 4-D.C.R.D critical graph, then any minimum D.C.R.D set of G can have at most two vertices of degree less than or equal to $(p - 3)$.

Theorem 3.8: If G is a 4-D.C.R.D critical graph, then

(i) G is self-centered of diameter 2.

(ii) G is a block.

Proposition 3.5: There exists no graph G for which both G and $\overline{G}$ are 4-D.C.R.D critical.

Theorem 3.9: If G is a 5-D.C.D type (I) critical graph with $p \geq 7$ and having any one of the following property, then G is 5-D.C.R.D type (I) critical (structure having every minimum D.C.R.D set is a path of length k).

(i) G has more than one central vertex.

(ii) G has a unique central vertex and for a peripheral node v,

$|N_1(v)| = 1$ and $|N_3(v)| \geq 3$ (or) $|N_1(v)| \geq 3$ and $|N_3(v)| = 1$ (or) $|N_1(v)| \geq 3$ and $|N_3(v)| \geq 3$.

Proof: If G is a 5-D.C.D type (I) critical graph, then by Lemma 3.3.9, G is of diameter 4 and G must have a unique pair of peripheral nodes $(v, \overline{v})$. Also, G is of the structure given in Figure 3.6 and we have the following:

(a) <A>, <B> and <C> are cliques, where $A = N_1(v)$, $B = N_2(v)$ and $C = N_3(v)$

(b) $<A \cup B>$ and $<B \cup C>$ are cliques

(i) If G has more than one central vertex, then $|B| \geq 2$. Also, any D.C.D set D of G must contain exactly one vertex of B. Therefore, V – D has at least one vertex of B, say u. Then, u is adjacent to all the vertices of V – D (as every vertex in B is adjacent to all the vertices of (V(G) – {v, $\bar{v}$ }). Hence, any D.C.D set of G is a D.C.R.D set of G. Also addition of an edge between any two non-adjacent vertices in G will reduce the D.C.D number to 4 and it will not affect the restrained property. Hence, G is 5-D.C.R.D type (I) critical.

(ii) Let G be a 5-D.C.D type (I) critical graph with a unique central vertex having |A| = 1 and |C| ≥ 3 (or) |A| ≥ 3 and |C| = 1 (or) |A| ≥ 3 and |C| ≥ 3. That is, G can have at most one pendant vertex. Also any D.C.D set D of G must contain exactly one vertex from each A and C, say x and y. Thus, V – D has at least two vertices from the set A (or) C (or) both A and C. Also <V – D> has at most two components A – {x} and C – {y} and they each is a clique of size greater than or equal to 2. Hence, every vertex in V – D is adjacent to at least one vertex of that component and hence any D.C.D set of G is a D.C.R.D set of G. Also addition of an edge between any two non-adjacent vertices in G will reduce the D.C.D number to 4 and it will not affect the restrained property. Hence, G is 5-D.C.R.D type (I) critical.

Remark 3.3: As any 5-D.C.D type (I) critical graph is 5-D.C.R.D type (I) critical, we have the following theorem without proof.

Theorem 3.10: If G is a 5-D.C.R.D type (I) critical graph, then

1. G is of radius 2 and diameter 4.
2. G need not be a block.
3. G is diameter edge (addition) critical.
4. G is Hamiltonian, if G is a block.

Proposition 3.6: Any 5-D.C.R.D type (I) critical graph with p ≥ 7 has at most two cut vertices.

Proof: Let G be a 5-D.C.R.D type (I) critical graph and let D = {v, x, y, z, $\bar{v}$ } be a minimum D.C.R.D set of G. Then, clearly <D> is the diametral path of length 4. Hence, <D> has three cut vertices, one is central vertex y of eccentricity 2 and the other two cut vertices are x and z of eccentricities 3. Hence, G can have at most 3 cut vertices namely x, y and z. Since, G is 5-D.C.R.D type (I) critical G must have any one of the following property.

Case (i): ***G has more than one central vertex***

If G has more than one central vertex, then y cannot be a cut vertex. Hence, the vertices x and z can only be the cut vertices of G and hence G can have at most two cut vertices.

Case (ii): ***G has a unique central vertex and for a peripheral node v, $|N_1(v)| = 1$ and $|N_3(v)| \geq 3$(or) $|N_1(v)| \geq 3$ and $|N_3(v)| = 1$ (or) $|N_1(v)| \geq 3$and $|N_3(v)| \geq 3$***

If G has a unique central vertex namely y, then y should be a cut vertex of G. Also,

(a) If $|N_1(v)| = 1$ and $|N_3(v)| \geq 3$, then x is the cut vertex of G and similarly if $|N_1(v)| \geq 3$ and $|N_3(v)| = 1$, then z is the cut vertex of G. Hence, G must have exactly two cut vertices, namely y and x (or z).

(b) If $|N_1(v)| \geq 3$ and $|N_3(v)| \geq 3$, then any vertex $x \in N_1(v)$ and $z \in N_3(v)$ cannot be a cut vertex of G as <N_1(v)>, <N_3(v)> are cliques in G. Hence, G must have exactly one cut vertex, namely y.

Hence if G is a 5-D.C.R.D type (I) critical graph, then G can have at most two cut vertices.

Corollary 3.4: Any 5-D.C.R.D type (I) critical graph has at most two pendant vertices.

Proposition 3.7: If u is a cut vertex of a 5-D.C.R.D type (I) critical graph G, then

(i) G – u has exactly two components and;

(ii) At least one component of G – u is a clique.

Proposition 3.8: If u is a cut vertex of a 5-D.C.R.D type (I) critical graph, then $|N_3(u)| = 1$ or $|N_2(u)| = 2$.

Proof: Let u be a cut vertex of a 5-D.C.R.D type (I) critical graph G and let (v, $\bar{v}$) be the unique pair of peripheral nodes of G. Then we have the following two cases.

Case (i): u is adjacent to a pendant vertex

In this case e(u) = 3 and u is adjacent to one of the peripheral node say $\bar{v}$. Then clearly, u is adjacent to all the central vertices of G. Hence, $N_3(u) = \{v\}$ and hence $|N_3(u)| = 1$.

Case (ii): u is a central vertex of G

In this case e(u) = 2 and u is non-adjacent to only the peripheral nodes (v, $\bar{v}$) of G. Hence, $|N_2(u)| = 2$.

Proposition 3.9: Let G be a 5-D.C.R.D type (I) critical graph with two cut vertices x and y. Then, $|N_3(x) \cup N_3(y)| \leq 2$.

Proof: Let G be a 5-D.C.R.D type (I) critical graph with two cut vertices say x and y. Then we have the following two cases.

Case (i): Both x and y are adjacent to a pendant vertex

Then, by previous proposition $|N_3(x)| = 1$ and $|N_3(y)| = 1$. Hence, $|N_3(x) \cup N_3(y)| \leq 2$.

Case (ii): x is a central vertex and y is adjacent to a pendant vertex.

Then by previous proposition, $|N_2(x)| = 2$, $|N_3(x)| = 0$ and $|N_3(y)| = 1$. Hence, $|N_3(x) \cup N_3(y)| = 1$.

Therefore if x and y are any two cut vertices of a 5-D.C.R.D type (I) critical graph G, then $|N_3(x) \cup N_3(y)| \leq 2$.

Proposition 3.10: Let G be a 5-D.C.R.D type (I) critical graph. If G has exactly one pendant vertex and one central vertex, then

$$q = \binom{p-3}{2} + (p-2).$$

Proof: Let w be a support of a pendant vertex $\bar{v}$ in G. Then, $|N_3(w)| = 1$. Let u be the central vertex of G. Thus, u and $\bar{v}$ are the two vertices which are adjacent to w. Also, u and w are the two cut vertices of G. Then, we have the following:

(i) d(u) = p – 3 and

(ii) $\langle V(G) - \{u, w, \bar{v}\}\rangle$ is a clique.

Hence, $q = \binom{p-3}{2} + (p-3) + 1$ (where 1 represents the edge $w\bar{v}$)

$$\Rightarrow q = \binom{p-3}{2} + (p-2).$$

Remark 3.4: The following structures of 5-D.C.R.D type (I) critical graphs are having exactly one pendant vertex and one central vertex. Also, the above bound is attainable for these structures of graphs.

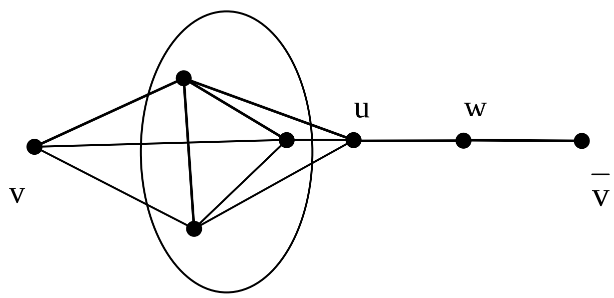

Figure 3.2 - ***Structure of a 5-D.C.R.D type (I) critical graph having exactly one pendant vertex and one central vertex***

Proposition 3.11: Let G be a 5-D.C.R.D type (I) critical graph with two pendant vertices, then

$$q = \binom{p-4}{2} + 2(p-3).$$

Proof: Let u and w be the support of the two pendant vertices v and $\bar{v}$ respectively. Since G is of diameter 4 and G has two pendant vertices, u and w are the only two vertices with eccentricity 3 in G.

Hence, all the vertices of <V(G) – {v, u, w, $\bar{v}$}> are the central vertices of G and hence <V(G) – {v, u, w, $\bar{v}$}> is a clique. Also, the vertices u and w must be adjacent to all the central vertices of G as they are vertices of eccentricity 3.

Hence, $q = \binom{p-4}{2} + 2(p-4) + 2$ (where 2 represents the edges vu and w$\bar{v}$)

$$q = \binom{p-4}{2} + 2(p-3).$$

Remark 4.3.5: The following structures of 5-D.C.R.D critical graphs are having two pendant vertices. Also, the above bound is attainable for these structures of graphs. Here, C is the set of central vertices with |C| ≥ 3 and <C> is a clique.

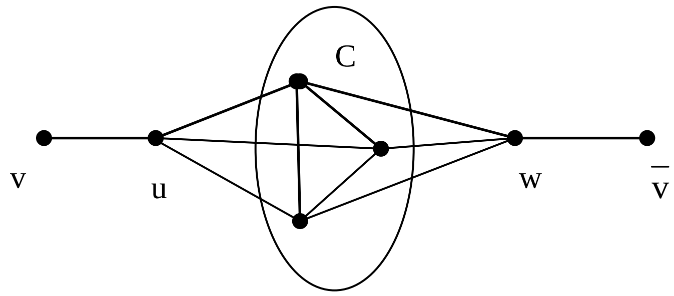

Figure 3.3 - ***Structure of a 5-D.C.R.D type (I) critical graph having two pendant vertices and more central vertices***

Proposition 3.12: Let G be a 5-D.C.R.D critical graph without pendant vertex. If G has exactly one central vertex, then

$$q = \binom{p-(n+1)}{2} + \binom{p-(m+1)}{2} + (m+n-2), \text{ where } m, n \geq 4 \text{ and } m + n = p - 1.$$

Proof: Let G be a 5-D.C.R.D critical graph without pendant vertex and let u be the central vertex of G. Since u is the only central vertex of G, u is the cut vertex of G. Thus, G – u has two components C_1 and C_2 such that each component is a clique and $|C_1| \geq 4$ and $|C_2| \geq 4$. Let $C_1 = K_m$, $m \geq 4$ and $C_2 = K_n$, $n \geq 4$. Then, m + n + 1 = p (where 1 represents the vertex u). Also $|N_2(u)| = 2$. That is, u must be non-adjacent to exactly one vertex of each component K_m, $m \geq 4$ and K_n, $n \geq 4$ of G – u. Therefore, u must be adjacent to (m – 1) vertices of K_m, $m \geq 4$ and (n – 1) vertices of K_n, $n \geq 4$.

Hence, $q = \binom{p-(n+1)}{2} + \binom{p-(m+1)}{2} + (m-1) + (n-1),$ where m, n ≥ 4

$$q = \binom{p-(n+1)}{2} + \binom{p-(m+1)}{2} + (m+n-2), \text{ where } m, n \geq 4 \text{ and } m + n = p - 1.$$

Remark 3.6: The following structures of 5-D.C.R.D type (I) critical graphs are without pendant vertices and have exactly one central vertex. Also the above bound is attainable for these structures of graphs.

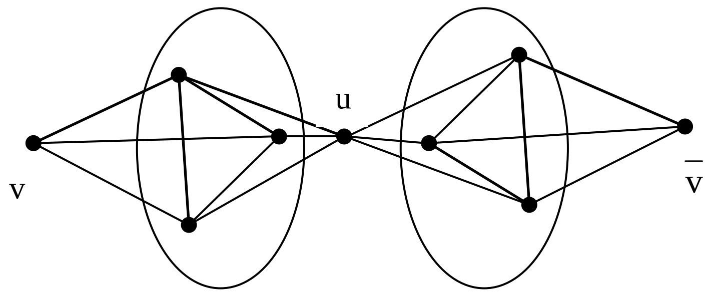

Figure 3.4 - ***Structure of a 5-D.C.R.D type (I) critical graph without pendant vertices and has exactly one central vertex***

Theorem 3.11: Any 6-D.C.D type (II) critical graph with $p \geq 8$ is 6-D.C.R.D type (II) critical (structure having every minimum D.C.R.D set is a cycle of length k).

Proof: Let G be a 6-D.C.D type (II) critical graph with $p \geq 8$ and let D be a minimum D.C.D set of G. Then G is of the structure given in Figure 3.7 and we have the following.

(i) For every vertex $v \in V(G)$, $<N_1(v)>$ has two components A_1 and A_2 such that $<A_1>$ and $<A_2>$ are cliques and similarly $<N_2(v)>$ has two components B_1 and B_2 such that $<B_1>$ and $<B_2>$ are cliques.

(ii) Also, $<A_1 \cup B_1>$ and $<A_2 \cup B_2>$ are cliques.

(iii) $<N_3(v)>$ is a clique.

Claim: ***Every D.C.D set of G is a D.C.R.D set***

Since G is 3-self-centered, every vertex of G must lie on a C_6 and every C_6 is a D.C.D set of G. Let v be a vertex with degree $\delta \geq 2$ in G. Then $|N_2(v)| \geq 3$ (as $p \geq 8$) and any D.C.D set D of G must contain exactly two vertices $\{b_1, b_2\}$ of $N_2(v)$. Hence, every vertex in V – D is adjacent to at least one vertex of $N_2(v) - \{b_1, b_2\}$ and hence D becomes a D.C.R.D set of G.

Let v be a vertex with $d(v) \geq 3$ in G and let x and y be any two non-adjacent vertices of G. Then we have the following cases:

(i) $x, y \in N_1(v)$;

(ii) $x, y \in N_2(v)$ and ;

(iii) $x \in N_1(v)$, $y \in N_2(v)$ and vice versa.

In all the cases, there exists a minimum D.C.D set that contains both x and y. Let $D = \{v, a_1, b_1, \bar{v}, b_2, a_2\}$, where $a_1 \in A_1$, $a_2 \in A_2$; $b_1 \in B_1$, $b_2 \in B_2$ and $\bar{v} \in N_3(v)$ (refer Figure 3.7) be a minimum D.C.D set of G. Then $(x, y) = (a_1, a_2)$ or $(x, y) = (b_1, b_2)$ or $(x, y) = (a_1, b_2)$ or $(x, y) = (a_2, b_1)$. Now, if we add an edge between x and y, then in all the cases the set of vertices $D^1 = \{a_1, a_2, b_1, b_2\}$ will form a D.C.D set for (G + xy). Also, any vertex in $V - D^1$ is adjacent to either v or $\bar{v}$. In particular, every vertex in $N_1(v) - \{a_1, a_2\}$ is adjacent to v (as $|N_1(v)| \geq 3$) and every vertex in $N_2(v) - \{b_1, b_2\}$ is adjacent to $\bar{v}$ (as $|N_2(v)| \geq 3$). Hence, D^1 becomes a D.C.R.D set of (G + xy) and hence G is 6-D.C.R.D type (II) critical.

Remark 3.7: As any 6-D.C.D type (II) critical graph is 6-D.C.R.D type (II) critical, we have the following theorems without proof.

Theorem 3.12: If G is a 6-D.C.R.D type (II) critical graph, then

(i) G is self-centered of diameter 3.

(ii) G is a block.

Corollary 3.5: If G is a 6-D.C.R.D type (II) critical graph, then $\Delta(G) \leq p - 4$ and $\delta(G) \geq 2$.

Theorem 3.13: Any 6-D.C.R.D type (II) critical graph is Hamiltonian.

Theorem 3.14: If G is a 6-D.C.R.D type (II) critical graph, then G is radius edge critical.

3.1.2 Generalization of k-D.C.R.D type (I) and type (II) critical graphs
Following are some generalized results of k-D.C.R.D type (I), (when k is odd) and k-D.C.R.D type (II), (when k is even) critical graphs for $k \geq 5$.

Theorem 3.15: Any k-D.C.D type (I) critical graph, which is also a block, is k-D.C.R.D type (I) critical.
Theorem 3.16: If G is a k-D.C.R.D type (I) critical graph, then we have the following:

(i) G is of diameter $k - 1$ and radius $\frac{k-1}{2}$.
(ii) G can have at most 2 cut vertices.
(iii) G can have at most 2 pendant vertices.

Theorem 3.17: Any k-D.C.R.D type (I) critical graph is diameter edge (addition) critical.

Theorem 3.18: Any k-D.C.R.D type (I) critical graph, which is also a block, is Hamiltonian.
Proof: For any vertex v of a k-D.C.R.D type (I) critical graph G, $\langle N_i(v) \rangle$ and $\langle N_i(v) \cup N_{i+1}(v) \rangle$, i = 1 to $k - 1$ are cliques and also $|N_i(v)| \geq 2$, where i = 1 to $k - 1$. Hence, there exists a cycle that covers all the vertices of G and hence G is Hamiltonian.

Theorem 3.19: Any k-D.C.D type (II) critical graph is k-D.C.R.D type (II) critical.

Theorem 3.20: If G is a k-D.C.R.D type (II) critical graph, then we have the following:

(i) G is $\frac{k}{2}$-self-centered. (ii) G is a block.

Theorem 3.21: Any k-D.C.R.D type (II) critical graph is radius edge (addition) critical.
Proof: As any k-D.C.R.D type (II) critical graph G is $\frac{k}{2}$-self-centered, every pair of non-adjacent vertices must lie on a same C_k. Thus, addition of an edge between every pair non-adjacent vertices will reduce the eccentricities of some vertices to $\frac{k}{2} - 1$ and at least one pair of vertices has the same eccentricity $\frac{k}{2}$. Hence, G must be radius edge critical.
Theorem 3.22: Any k-D.C.R.D type (II) critical graph is Hamiltonian.

3.2 k-D.C.R.D critical graphs due to restrained property
From the structure of a star, we can obtain the general structure of a k-D.C.R.D critical graph (due to restrained property) G with radius 1 and diameter 2. Also, addition of every edge e in G will reduce the D.C.R.D number of G to $k - 1$ or $k - 2$ and $(G + e)$ has the same radius and diameter as that of G.

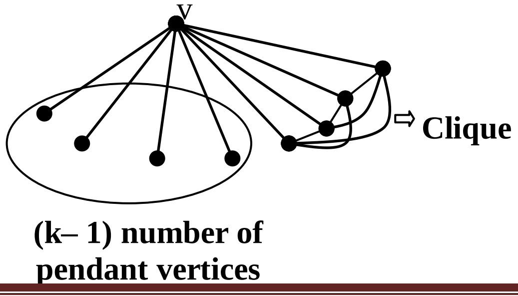

Figure 3.6 - *Structure of k-D.C.R.D critical graphs due to restrained property*

In this structure, $\gamma_{dcl} = 3$. Since, we have (k – 1) number of pendant vertices, each must be in the D.C.R.D set. Thus the vertex v together with the (k – 1) pendant vertices will form a D.C.R.D set and $\gamma_{rdcl} = k$. Also, addition of every edge will reduce the D.C.R.D number to (k – 1) or (k – 2). Hence, the above structures of graphs are k-D.C.R.D critical.

4. CONCLUSION

In this paper, the critical concept of distance closed restrained domination with respect to both distance closed and restrained property are analyzed and the structural properties of k-D.C.R.D critical graphs such as radius, diameter, number of cut vertices, clique components and Hamiltonian properties are studied. Since this concept deals the reduction in the cardinality of distance closed restrained dominating set for any addition of one new link in the original structure, it will be useful to study the communication model, which reduces it dominating parameters by simple addition of a link, which doesn't exist in the system. Hence this critical concept can be directly applied to the construction of a fault tolerant communication model.

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